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Axisymmetric rotating flow past a circular disk

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The steady, inviscid, axisymmetric, rotating flow past a circular disk in an unbounded liquid is determined on the hypothesis that all streamlines originate in a uniform flow far upstream of the body. The characteristic parameter for the flow is $k = 2\Omega a/U$, where Ω and U are the angular and axial velocities of the basic flow and a is the radius of the disk. Forward separation is found to occur for $k > k_* = 1{\cdot}9$, in agreement with observation (Orloff & Bossel 1971). The length of the upstream separation bubble is determined on the hypothesis that the previous solution remains valid for $k > k_*$ despite the existence of closed streamlines within the upstream separation bubble (which may, but do not necessarily, invalidate the solution). This length increases rapidly for $k > 3$, in qualitative agreement with observation. The hypothesis of unseparated flow implies a singularity at the rim of the disk, just as in potential flow. The strength of this singularity departs only slightly from its potential-flow value for $0 \leqslant k \leqslant 2$, but increases rapidly with k for $k > 3$, which suggests that (quite apart from the difficulties implied by the existence of closed streamlines) the solution cannot remain valid for sufficiently large k.

1. Introduction

This paper aims at a theoretical prediction, on the basis of an inviscid model, of the conditions for the formation and growth of a separation bubble upstream of an obstacle in a steady, axisymmetric, rotating flow. The characteristic parameter is the inverse Rossby number

$$k = 2\Omega a/U, \tag{1.1}$$

where Ω and U are the rotational and translational velocities of the basic flow and a is the maximum transverse radius of the obstacle. Theoretical considerations suggest that rotation decreases the stagnation-point acceleration by an amount that increases with k, such that forward separation occurs for $k > k_*$ and leads to the formation of an upstream separation bubble of length $s(k)$. They also suggest that $k_* = k_*(T)$ and $s = s(k, T)$ in a real fluid of viscosity ν, where

$$T = \Omega a^2/\nu \equiv \tfrac{1}{2}kRe \equiv 1/E \tag{1.2}$$

is the Taylor number, Re is the Reynolds number, and E is the Ekman number. The double limit $k \uparrow \infty$, $T \uparrow \infty$, corresponding to slow inviscid motion, yields a Taylor column, $s \uparrow \infty$, inside which the axial velocity vanishes. The double limit $k \downarrow 0$, $Re \uparrow \infty$ yields potential flow, for which $k_* = s = 0$.

The only quantitative observations of upstream separation bubbles appear to be those of Maxworthy (1970) for a sphere and those of Orloff & Bossel (1971) both for a sphere and for a disk with a conical afterbody. Their experiments differed in at least three significant respects: (i) Maxworthy did not (during the observations cited herein) impose any constraints against, whereas Orloff & Bossel prevented, axial rotation of the obstacle; (ii) Maxworthy's spheres were significantly smaller, with $a/R = 0{\cdot}04$ versus $0{\cdot}24$ for Orloff & Bossel (where R is the radius of the tube that confined the rotating flow); (iii) Maxworthy varied k and T independently, whereas Orloff & Bossel did not.

Maxworthy reports forward separation for all $k \gtrsim 1$, and possibly for k as small as $0{\cdot}5$, that $s(k, T)$ is a non-decreasing function of k that tends asymptotically to a finite limit, say $\tilde{s}(T)$, and that $\tilde{s}(T)$ increases linearly with T for $T > 200$. His limiting results imply that s increases exponentially with k for sufficiently large T and suggest that it may be independent of T for sufficiently small k. His data are inadequate for the determination of k_*, but they do imply $k_* \lesssim 1$ and suggest the possibility that forward separation could occur for any positive value of k. Maxworthy describes the separation bubble or "slug" as "a sharply defined region of almost stagnant fluid [in the sense that] the axial velocity is very low, although the relative swirl velocity can be quite large"; he does not report any recirculating flow within the bubble. He also reports that the sphere did not rotate for $k \lesssim 1$, rotated at an increasing rate for $1 \lesssim k \lesssim 5$ and rotated with the basic flow for $k \gtrsim 5$.

Orloff & Bossel report that forward separation from a sphere does not occur for $k \lesssim 2{\cdot}6$, that s increases linearly with k for $4{\cdot}6 \lesssim k \lesssim 8{\cdot}4$ and that a Taylor column ($s = \infty$) forms for $k \gtrsim 9$; however, they did not actually measure k_* for the sphere (the value $k_* = 2{\cdot}6$ was inferred from a rather good, straight-line fit to thirteen points for s/a *vs.* k in $4{\cdot}6 < k < 8{\cdot}4$), and the significance of their description "Taylor column" is limited by the finite length of their apparatus (with $s \simeq 30a$ as the maximum observable value or bubble length). For a disk of radius a with a conical afterbody of length a, they report $k_* \approx 1{\cdot}9$ (directly measured) and Taylor-column formation for $k \gtrsim 10$. They report recirculating flow within the separation bubble and a definite stagnation point at its upstream apex.

A theoretical calculation for a sphere in an unbounded inviscid flow ($a/R \downarrow 0$, $T \uparrow \infty$) yields $k_* = 2{\cdot}2$ on Long's hypothesis of uniform upstream flow (Miles 1971). This is close to the value ($k_* \simeq 2{\cdot}6$) reported by Orloff & Bossel but far from that ($k_* \lesssim 1$) implied by Maxworthy's observations. It therefore appears to be significant that the theoretical model implies a non-rotating disk, which corresponds more closely to the former observations. This same theoretical calculation also yields a downstream shift of the aft separation ring with increasing k; in this instance, however, the maximum calculated shift of 24° is in qualitative agreement with Maxworthy's observations, whereas Orloff & Bossel report that the flow is unseparated (corresponding to a rotation-induced shift of roughly 90° for the aft separation ring) for sufficiently large k.

The theoretical question as to whether there exists a critical value of k for the formation of a true Taylor column in an inviscid flow remains controversial.

It does appear, however, that there exists a representative value of k, say k_T, at which the length of the upstream separation bubble begins to increase rapidly with k if the Taylor number is sufficiently large. A calculation of the asymptotic (as $t \to \infty$) speed at which the stagnation point moves forward from an impulsively started disk in the slow-motion approximation, $U = 0{\cdot}675\Omega a$ (Greenspan 1968, § 4.3), yields $k_T = 3{\cdot}0$. This calculation is significant in showing that a forward stagnation point can move upstream, but the finite value of k_T is inconsistent with the slow-motion approximation (which is strictly valid only for $k \uparrow \infty$) on which it is based; moreover, the model does not permit the stagnation point to remain at equilibrium in a steady flow.

Against this background, we consider steady, laminar, inviscid, axisymmetric, rotating flow past a circular disk in an unbounded liquid on the basis of Long's hypothesis. [The calculation of $s(k)$ for a sphere would be more difficult than for the disk; moreover, the known results for the disk in the limit $k \uparrow \infty$ (Greenspan 1968) provide a valuable basis of comparison.] This is, to be sure, a highly idealized model and is open to the following objections: (i) Long's hypothesis may fail in consequence of any or all of transient, viscous and boundary effects (the limits $t \uparrow \infty$, $\nu \downarrow 0$ and $R \uparrow \infty$ may be unrealistic and/or not interchangeable). (ii) A real flow must separate at the rim of the disk, and the resulting downstream wake may affect the upstream flow. (iii) Long's hypothesis, as actually invoked in the derivation of (2.2) below, implies not only that the upstream flow is uniform, but also that all streamlines originate in this uniform flow. In fact, all streamlines are closed within the upstream separation bubble, in consequence of which the predicted flow within, and the shape of, the bubble may be (but are not necessarily) incorrect. We deal with these objections in turn.†

(i) Long's hypothesis, which precludes the formation of a true Taylor column, has been the subject of much controversy (see Greenspan's remarks, 1968, § 4.6), partially because the slow-motion ($k \uparrow \infty$) model predicts Taylor columns for either $t \uparrow \infty$ or $\nu \downarrow 0$ (Greenspan 1968, § 4.3), and partially because of the implicit existence of an upstream impulse proportional to the lee-wave drag (Benjamin 1970). These difficulties are resolved, at least in principle, by McIntyre's (1972) discovery that second-order transient effects *downstream* of an obstacle in a bounded (by a circular tube) rotating flow give rise to cylindrical waves that propagate upstream of the obstacle and appear as a columnar disturbance in the limit $t \uparrow \infty$. McIntyre also shows that these second-order effects are evanescent in an unseparated unbounded flow, and an extension of his calculation yields the asymptotic approximation (Miles 1972)

$$u_0/U \sim 0{\cdot}01 F_1^2 k^5 (a/R) \quad (2\Omega R/U \uparrow \infty) \tag{1.3}$$

for the columnar velocity u_0 upstream of a small obstacle of dimensionless dipole moment F_1 ($F_1 \to 1$ for a sphere and $4/(3\pi)$ for a disk as $k \downarrow 0$; see figure 3 below) in a tube of radius R. This result, which is based on a perturbation calculation, is valid only for $u_0/U \ll 1$ and depends essentially on the assumption of an inviscid fluid [McIntyre shows that a true columnar disturbance is impossible in

† The phenomena cited in (i)–(iii) are not necessarily independent. For example, closed streamlines may occur downstream, as well as upstream, of the disk.

a bounded, viscous flow in the limit $T \uparrow \infty$; that the same result holds for an unbounded flow follows from a calculation similar to that given by Miles (1970) for an oseenlet]. Nevertheless, (1.3) suffices to show that upstream influence is negligible for sufficiently small a/R and moderate k, but that it may ultimately be significant as $k \uparrow \infty$ with a/R fixed. Considering, for example, Maxworthy's experiments with half-inch spheres in a twelve-inch tube, we obtain $u_0/U = (0{\cdot}01, 0{\cdot}1, 0{\cdot}4, 1{\cdot}3)$ for $k = (2, 3, 4, 5)$; these same values of u_0/U are roughly applicable to Orloff & Bossel's disk but must be multiplied by six for their sphere.

(ii) It is widely accepted that the effects of downstream separation on the upstream flow are only secondary if $k = 0$—for example, the observed, anterior flow over a sphere at high Reynolds numbers is qualitatively similar to that predicted on the hypothesis of potential flow, even though separation actually occurs near the equator. A rather successful comparison between the theoretically predicted (Miles 1969*a*) and observed (Maxworthy 1970) flows upstream of the forward separation bubble in a rotating flow, wherein the bubble is incorporated into the theoretical model by regarding it as part of an equivalent obstacle, suggests that the downstream wake has only a secondary effect on the upstream flow for moderate k and small a/R (the aforementioned comparisons were for k between 2 and 3 and $a/R = 0{\cdot}04$). On the other hand, this may not be true in that regime of k and a/R in which nonlinear interactions are not small, especially as these interactions are initiated in the downstream flow. [McIntyre (1972) has pointed out that resonant interactions among the lee waves could lead to secular instability of the lee-wave train. Such an instability could yield a downstream wake even in the absence of viscous separation and also could diminish the strength of the cylindrical waves that propagate upstream of the obstacle.] A comparison between theory (Miles 1969*b*) and experiment (Pritchard 1969) for accelerated flow past a sphere also supports the contention that the downstream wake has only a secondary effect on the upstream flow, although in this case acceleration may inhibit downstream separation. It may be significant that separation occurs at the rim of the disk independently of k, whereas the position of the separation ring on the sphere depends on k; this suggests that the upstream and downstream flows may be more effectively decoupled for a disk than for a sphere.

There is, of course, no question as to the importance of separation for the downstream flow. Stewartson (1968) suggests that a more realistic prediction of that flow may be obtained by admitting a columnar disturbance upstream of the obstacle and determining its strength through the requirement that the tangential velocity should vanish on the posterior surface of the obstacle. Trustrum (1971) shows that such a model (which, following Stewartson, she calls an Oseen model) does provide a theoretical prediction of the downstream flow that is qualitatively similar to observation for stratified flow over a thin barrier (the two-dimensional analogue of the circular disk); however, the predicted upstream flow differs significantly from observation.

(iii) The inviscid flow predicted on Long's hypothesis is a valid solution of the equations of motion even if closed streamlines do appear in the flow; however, the solution then is not unique and is likely to be at least locally unstable. There seems to be little doubt that viscosity exerts a significant influence on the flow inside the separation bubble; on the other hand, the aforementioned comparison between theory and experiment supports the validity of both an inviscid model and Long's hypothesis for the exterior flow. The essential question, then, is whether the probable failure of the basic model to provide an adequate description of the flow inside the bubble also implies its failure to provide an adequate approximation to the shape, or at least the length, of the bubble. This question is perhaps best answered by carrying out the calculation and comparing the results with observation.

The boundary-value problem for a circular disk in a rotating flow may be solved either through separation of variables in oblate spheroidal co-ordinates or through the construction and approximate solution of an integral equation for an equivalent-vortex-sheet density. We give the former solution in § 3 and the latter formulation in § 5 (but this solution also has been carried out in detail). The predicted value of k_* is 1·9, in agreement with the value measured by Orloff & Bossel. The results for $s(k)$ are qualitatively consistent with their observations in that they imply a rapid growth of $s(k)$ for $k > 3$. They also imply a rapid increase of the strength of the singularity at the rim of disk for $k > 3$, in consequence of which the inviscid model becomes increasingly less realistic. Further theoretical progress would appear to demand a model that incorporates viscous effects, at least in the neighbourhood of the rim, and probably throughout the upstream separation bubble.

2. Boundary-value problem

Referring all lengths and velocities to a and U, respectively, we derive the velocity from a vector potential according to

$$\mathbf{v} = \nabla \times \boldsymbol{\phi} + k\boldsymbol{\phi}, \quad \boldsymbol{\phi} = \boldsymbol{\phi}_1 \phi(x, r), \tag{2.1}$$

where $\boldsymbol{\phi}_1$ is a unit vector in the azimuthal direction of rotation, $r\phi$ is the Stokes stream function and r is the cylindrical radius. The scalar ϕ satisfies:

$$\phi_{xx} + \phi_{rr} + r^{-1}\phi_r + (k^2 - r^{-2})\,\phi = \tfrac{1}{2}k^2 r \tag{2.2}$$

[cf. Batchelor (1967, § 7.5), in whose notation $\psi = r\phi$, $C = k\psi$ and $dH/d\psi = \frac{1}{2}k^2$], where x and r are cylindrical polar co-ordinates; the boundary condition (of tangential flow on the disk)

$$\phi = 0 \quad (x = 0, \quad 0 \leqslant r \leqslant 1); \tag{2.3}$$

and the radiation condition (no inertial waves appear in the upstream flow)

$$\phi \sim \tfrac{1}{2}r + o(1/kR) \quad (kR \to \infty, x < 0), \tag{2.4}$$

where R is the spherical radius.

3. Spheroidal co-ordinate solution

We pose ϕ in the form

$$\phi = \tfrac{1}{2}r - \tfrac{1}{2} \sum_{n=1}^{\infty} B_n \phi_n(\xi, \eta), \tag{3.1}$$

where $\frac{1}{2}r$ is a particular solution of (2.2) that yields the basic flow, ξ and η are oblate spheroidal co-ordinates, defined by

$$x = \xi\eta, \quad r = (\xi^2+1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}} \quad (\xi \geqslant 0, \ -1 \leqslant \eta \leqslant 1), \tag{3.2a, b}$$

the ϕ_n are a complete set of complementary solutions of (2.2), each of which satisfies (2.4) or, equivalently,

$$\phi_n = o(1/k\xi) \quad (k\xi \to \infty, \ -1 < \eta < 0), \tag{3.3}$$

and the B_n are determined by (2.3), which implies

$$\sum_{n=1}^{\infty} B_n \phi_n(0,\eta) = (1-\eta^2)^{\frac{1}{2}} \quad (-1 \leqslant \eta \leqslant 1). \tag{3.4}$$

Solving (2.2) by separation of variables, and proceeding as in the corresponding problem for stratified flow over a thin barrier (Miles 1968), we pose ϕ_n in the form

$$\phi_n(\xi,\eta) = G_n(\xi)\, S_n(\eta) + \sum_{m=1}^{\infty} D_{mn} F_m(\xi)\, S_m(\eta), \tag{3.5a}$$

where

$$F_n(\xi) \equiv R^{(1)}_{1n}(-ik, i\xi), \tag{3.5b}$$

$$G_n(\xi) \equiv R^{(2)}_{1n}(-ik, i\xi)/R^{(2)}_{1n}(-ik, i0), \tag{3.5c}$$

$$S_n(\eta) \equiv S_{1n}(-ik, \eta), \tag{3.5d}$$

$R^{(1,2)}_{1n}$ and S_{1n} are spheroidal wave functions in the notation of Flammer (1957), and the D_{mn} are to be determined by (3.3). We refer to equations in Flammer's monograph by the prefix F. Invoking the asymptotic approximations given by F(4.1.13, 20) for $R^{(1,2)}_{1n}$ and remarking that each of the odd ($n = 1, 3, \ldots$) and even ($n = 2, 4, \ldots$) subsets of the $S_n(\eta)$ is complete and orthogonal in $0 < \eta \leqslant 1$ and that

$$S_n(\eta) = \operatorname{sgn}\eta \sum_{m=1}^{\infty} (M_{mn}/N_m)\, S_m(\eta) \quad (0 < |\eta| \leqslant 1), \tag{3.6}$$

where

$$M_{mn} = \begin{cases} 2\displaystyle\int_0^1 S_m(\eta)\, S_n(\eta)\, d\eta & (n-m \text{ odd}) \quad (3.7a) \\ 0 & (n-m \text{ even}, \ m \neq n) \quad (3.7b) \end{cases}$$

and

$$N_m = 2\int_0^1 S_m^2(\eta)\, d\eta \tag{3.8}$$

($N_m \equiv N_{1m}$ in Flammer's notation), we find that (3.3) implies

$$D_{mn} = (-)^{\frac{1}{2}(n-m+1)} (M_{mn}/N_m)\, [R^{(2)}_{1n}(-ik, i0)]^{-1}. \tag{3.9}$$

Multiplying (3.4) through by $S_l(\eta)$, integrating over $(-1, 1)$, invoking the orthogonality of the S_l and the fact that $S_l(\eta)$ is even or odd in η according as l is odd or even, respectively, and remarking that

$$\phi_n(0,\eta) = S_n(\eta) \quad (n \text{ odd}), \tag{3.10}$$

we obtain

$$B_n = \begin{cases} \dfrac{1}{N_n}\displaystyle\int_{-1}^{1} (1-\eta^2)^{\frac{1}{2}}\, S_n(\eta)\, d\eta = \dfrac{4}{3}\dfrac{d_0^{1n}}{N_n} & (n \text{ odd}), \quad (3.11a) \\ 0 & (n \text{ even}), \quad (3.11b) \end{cases}$$

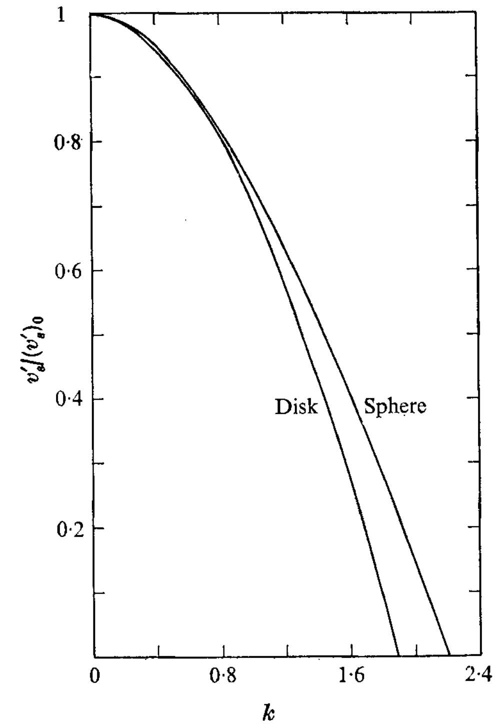

FIGURE 1. The stagnation-point velocity gradients for a disk and sphere relative to the respective limiting values for potential flow, $(v_s')_0 = 2/\pi$ and $\frac{3}{2}$.

where d_0^{1n} is the coefficient of $P_1^1(\eta)$ in the expansion of $S_n(\eta)$ in associated Legendre functions, see F(3.1.3*b*). It follows that n is odd and m is even throughout (3.4)–(3.9).

4. Velocity on disk

The velocity on the disk, as determined by (2.1) and (2.3), is purely radial. By invoking (3.1) and (3.5*a*), we obtain

$$v = |\eta|^{-1}(\partial\phi/\partial\xi)_{\xi=0} \tag{4.1a}$$

$$= |\eta|^{-1} \sum_{m=1}^{\infty} V_m S_m(\eta), \tag{4.1b}$$

where
$$V_m = -\tfrac{1}{2}\sum_{n=1}^{\infty} B_n[\delta_{mn} G_n'(\xi) + D_{mn} F_m'(\xi)]_{\xi=0}. \tag{4.2}$$

The radial derivative of the velocity at the stagnation point is given by

$$v_s' \equiv (dv/dr)_{r=0} = |\eta|^{-1}(1-\eta^2)^{\frac{1}{2}}(dv/d\eta)_{\eta=-1} \tag{4.3a}$$

$$= \sum_{m=1}^{\infty} V_m(-)^{m-1} c_0^{1m}(-ik), \tag{4.3b}$$

where c_0^{1m} is the coefficient of $(1-\eta^2)^{\frac{1}{2}}$ in the expansion of $S_m(\eta)$ in powers of $(1-\eta^2)$, see F(3.2.7).

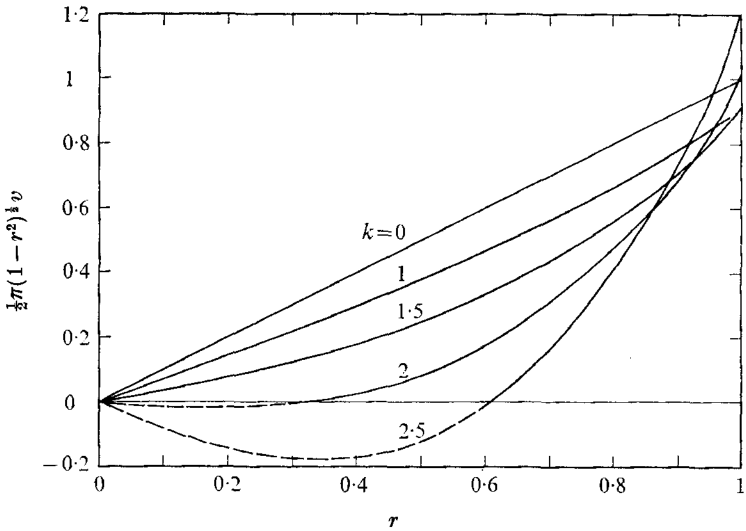

FIGURE 2. The normalized meridional velocity distribution on the upstream face of the disk. The dashed portions of the curves correspond to the reversed flow inside the stagnation bubble, where the present calculation is not expected to be quantitatively valid.

Retaining three terms in (4.3*b*) yields v_s' to three significant figures for $k \leqslant k_* = 1{\cdot}90$, at which point $v_s' = 0$ ($v_s' < 0$ for $k > k_*$). The result, together with the corresponding result for a sphere of radius a (Miles 1971), is presented in figure 1. The normalized (to remove the singularity at $r = 1$) velocity distribution is plotted in figure 2. The predicted position of the forward stagnation ring is $r_s = 0{\cdot}32$ and $0{\cdot}61$ for $k = 2$ and $2{\cdot}5$, respectively, and presumably tends to 1^- for $k \to \infty$.

5. Upstream flow

The axial perturbation velocity in the direction of motion of the disk relative to the fluid at rest is given by

$$u_0(x) = 1 - r^{-1}(r\phi)_r \quad (x < 0,\, r = 0) \tag{5.1a}$$

$$= 1 - (\xi^2 + \eta^2)^{-\frac{1}{2}}\{(1-\eta^2)\,\phi\}_\eta \quad (\xi = -x,\, \eta = -1) \tag{5.1b}$$

$$= (x^2+1)^{-\frac{1}{2}} \sum_{n=1}^{\infty} B_n\{G_n(|x|)\, c_0^{1n}(-ik) - \sum_{m=1}^{\infty} D_{mn} F_m(|x|)\, c_0^{1m}(-ik)\}. \tag{5.1c}$$

The expansion (5.1*c*) converges well in the neighbourhood of the disk, but only slowly for $|x| \gg 1$.

We obtain an alternative representation, which leads more directly to an asymptotic approximation for $|x| \gg 1$, in the form of a Fourier–Bessel integral. Introducing (cf. Miles 1968)

$$\zeta(r) = \tfrac{1}{2}\{\phi_x|_{x=0^+} - \phi_x|_{x=0^-}\} \tag{5.2a}$$

$$\equiv 0 \quad (r > 1) \tag{5.2b}$$

as the strength of an equivalent vortex sheet and the Hankel transform pair

$$Z(\beta) = \int_0^1 \zeta(r)\, J_1(\beta r)\, r\, dr, \quad \zeta(r) = \int_0^\infty Z(\beta)\, J_1(\beta r)\, \beta\, d\beta, \tag{5.3a,b}$$

we find that (2.2) and (2.4) are satisfied by

$$\phi(x,r) = \tfrac{1}{2}r + \left[2H(x)\int_0^k (k^2-\beta^2)^{-\frac{1}{2}} \sin\{(k^2-\beta^2)^{\frac{1}{2}} x\} \right.$$
$$\left. - \int_k^\infty (\beta^2-k^2)^{-\frac{1}{2}} \exp\{-(\beta^2-k^2)^{\frac{1}{2}}|x|\}\right] Z(\beta)\, J_1(\beta r)\, \beta\, d\beta, \tag{5.4}$$

where $H(x)$ is Heaviside's step function. The function $\zeta(r)$ is determined explicitly by (3.1), which, in conjunction with (3.5), yields

$$\zeta(r) = -(1-r^2)^{-\frac{1}{2}} \sum_1^\infty B_n G_n'(0)\, S_n\{(1-r^2)^{\frac{1}{2}}\}. \tag{5.5}$$

Alternatively, we may substitute (5.4) into (2.3) to obtain the integral equation

$$\int_k^\infty (\beta^2-k^2)^{-\frac{1}{2}} Z(\beta)\, J_1(\beta r)\, \beta\, d\beta = \tfrac{1}{2}r \quad (0 \leqslant r \leqslant 1), \tag{5.6}$$

which, together with (5.3), the constraint (5.2*b*), and the restriction that the singularity of $\zeta(r)$ at $r = 1$ be integrable, determines $\zeta(r)$; it may be solved approximately by Galerkin's method (cf. Miles 1968).

Integrating (5.4) twice by parts, we obtain

$$\phi \sim \tfrac{1}{2}r - \mathscr{A}_1 J_1(kr)\,(k|x|)^{-1} - \{k^2\mathscr{A}_3 J_1(kr) - \mathscr{A}_1 kr J_2(kr)\}\,(k|x|)^{-3}$$
$$+ O(|kx|^{-5}) \quad (kx \to -\infty), \tag{5.7}$$

where $\quad \mathscr{A}_1 = kZ(k) \equiv \tfrac{1}{2}k^2 F_1(k) \quad$ and $\quad \mathscr{A}_3 = k^{-1}(d/d\beta)\{\beta Z(\beta)\}|_{\beta=k} \quad$ (5.8*a*, *b*)

are the *upstream-influence parameters* of the disk, and F_1 is its *dipole moment* (cf. Miles 1969*a*). Substituting (5.5) into (5.3*a*), we obtain

$$Z(\beta) = -\sum_1^\infty B_n G_n'(0) \int_0^1 J_1\{\beta(1-\eta^2)^{\frac{1}{2}}\}\, S_n(\eta)\, d\eta. \tag{5.9}$$

The integral for $\beta = k$ may be evaluated with the aid of F(5.3.14) with the end result

$$\mathscr{A}_1(k) = -\tfrac{1}{2}\sum_{n=1}^\infty i^{1-n} B_n [R_{1n}^{(2)}(-ik, i0)]^{-1} S_n(0). \tag{5.10}$$

If $\beta \neq k$, the integral in (5.9) may be evaluated by introducing the expansion of $S_n(\eta)$ in $P_n^1(\eta)$ and using the known Hankel transform of P_n^1; the resulting expression for $\mathscr{A}_3(k)$ is complicated but numerically tractable in the range of interest. Numerical values of F_1 are compared with the corresponding results for a sphere (Miles 1971) in figure 3.

Substituting (5.7) into (5.1*a*), we obtain

$$u_0(x) \sim \mathscr{A}_1|x|^{-1} + \mathscr{A}_3|x|^{-3} + O(|x|^{-5}). \tag{5.11}$$

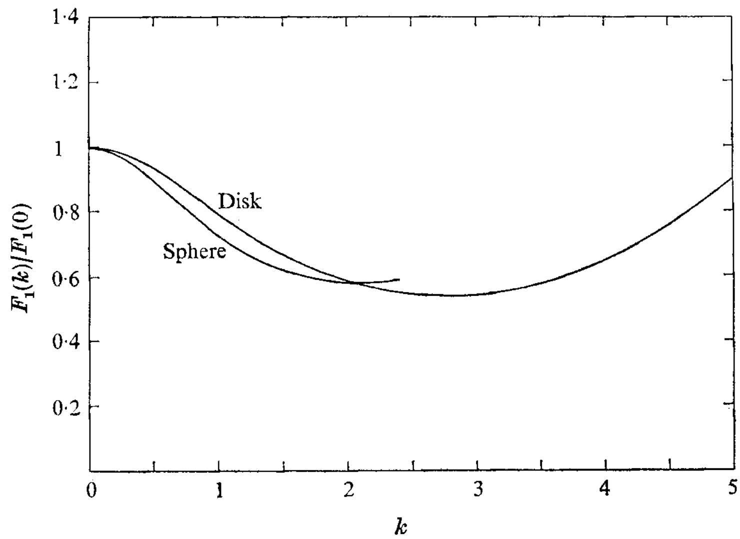

FIGURE 3. The dipole moment for a disk, as determined by (5.8 a) and (5.10), and for a sphere (Miles 1971). The respective, limiting values of F_1 for $k = 0$ (potential flow) are $4/(3\pi)$ and 1

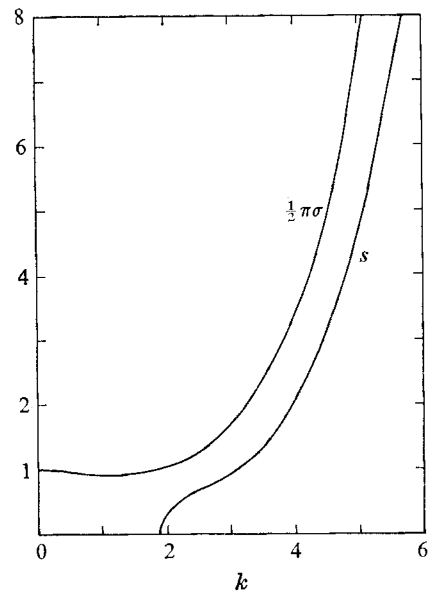

FIGURE 4. The location of the upstream stagnation point $x = -s(k)$, as defined by (5.12), and the strength of the rim singularity, as defined by (5.15).

The upstream stagnation point is determined by

$$u_0(x) = 1, \quad x = -s(k) < 0 \quad (k > k_*). \tag{5.12}$$

The confluence of this stagnation point with that on the disk ($x = 0^-$, $r = 0$) implies

$$s(k) \to C(k-k_*)^{\frac{1}{2}} \quad (k \downarrow k_*), \tag{5.13}$$

whilst (5.11) implies

$$s(k) \sim \mathscr{A}_1(k) \quad (k \uparrow \infty). \tag{5.14}$$

A numerical matching of (5.13) with the values of $s(k)$ implied by either (5.1c) or (5.11) yields $C = 0{\cdot}92$. The two-term approximation (5.11) provides an adequate approximation ($\pm 2\%$) to $s(k)$ for $k > 2{\cdot}6$, whilst (5.14) gives values accurate to within 1% for $k > 4$. The combined approximations are plotted in figure 4.

The preceding numerical results were obtained and checked by using Flammer's tabulated values of the various parameters and functions for $k = 1(0{\cdot}5)\,2{\cdot}5$, by the computation of these parameters and functions for $k \leqslant 6$ on a digital computer, and by a Galerkin solution of (5.6). Keeping only one (or four) term(s) in the expansion (5.10) yields $\mathscr{A}_1$ to within 1% for $k \leqslant 2$ (or 5.8). The numerical solution deteriorates rapidly for $k \geqslant 6$ in consequence of the rapid decrease of $R^{(2)}_{1n}(-ik, i0)$ with increasing k; this numerical difficulty could be overcome and the results extended to larger k, but the limited significance of the basic model for large k renders the effort unattractive. The available asymptotic results for the oblate spheroidal functions are inadequate for an asymptotic approximation for large k, but a heuristic investigation suggests that $\mathscr{A}_1$, and hence s, grows like e^{2k} as $k \uparrow \infty$.

The preceding solution implies that both the velocity and vorticity are singular at the rim according to

$$v \sim \zeta \sim \sigma(1-r^2)^{-\frac{1}{2}} \quad (r \uparrow 1), \tag{5.15a}$$

where

$$\sigma = -\tfrac{1}{2}\sum_{1}^{\infty} B_n G_n'(0)\, S_n(0) \tag{5.15b}$$

departs only slightly from its potential-flow value of $2/\pi$ for $k < 2$ but increases rapidly for $k > 3$; see figure 4. This suggests that viscous effects are comparable with those in a non-rotating flow for moderate k (including $k \leqslant k_*$), but that they become increasingly important, and our model correspondingly less realistic, with increasing k.

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